

Computational aspects of the higher Nash blowup of hypersurfaces

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Abstract

The higher Nash blowup of an algebraic variety replaces singular points with limits of certain spaces carrying higher-order data associated to the variety at non-singular points. In this note we will define a higher-order Jacobian matrix that will allow us to make explicit computations concerning the higher Nash blowup of hypersurfaces. Firstly, we will generalize a known method to compute the fiber of this modification. Secondly, we will give an explicit description of the ideal whose blowup gives the higher Nash blowup. As a consequence, we will deduce a higher-order version of Nobile's theorem for normal hypersurfaces.

Introduction

The main purpose of this note is to present, as the title suggests, some computational aspects of the higher Nash blowup of a hypersurface. The higher Nash blowup is defined as follows (see [No], [OZ], [Y]):

Let $X = \mathbf{V}(I) \subset \mathbb{C}^s$ be an irreducible algebraic variety of dimension d , given as the zero set of some ideal I . Let R be its ring of regular functions. For each $p \in X$, let (R_p, \mathfrak{m}_p) be the localization at p and define the $R_p/\mathfrak{m}_p \cong \mathbb{C}$ -vector space $T_p^n X := (\mathfrak{m}_p/\mathfrak{m}_p^{n+1})^\vee$. This is a vector space of dimension $D = \binom{d+n}{d} - 1$, whenever p is a non-singular point. The fact that $X \subset \mathbb{C}^s$ implies that $T_p^n X \subset T_p^n \mathbb{C}^s \cong \mathbb{C}^E$, where $E = \binom{s+n}{s} - 1$, that is, we can see $T_p^n X$ as an

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element of the Grassmanian $Gr(D, \mathbb{C}^E)$. Let $S(X)$ be the singular locus of X . Now consider the Gauss map:

$$G_n : X \setminus S(X) \rightarrow Gr(D, \mathbb{C}^E), \quad p \mapsto T_p^n X.$$

Denote by $Nash_n(X)$ the Zariski closure of the graph of G_n . Call π_n the restriction to $Nash_n(X)$ of the projection of $X \times Gr(D, \mathbb{C}^E)$ to X . When $n = 1$, the pair $(Nash_n(X), \pi_n)$ is usually called the *Nash modification* of X . For $n > 1$, $(Nash_n(X), \pi_n)$ is called the *higher Nash blowup* of X . This construction gives a canonical modification of an algebraic variety that replaces singular points by limits of sequences $\{T_{p_i}^n X\}$, where $\{p_i\} \subset X$ is any sequence of non-singular points converging to a singular one.

Unfortunately, despite of being a natural and geometrically attractive modification, it is hard to compute in general. The goal of this note is to deal with this problem, to some extent, in the case of hypersurfaces. We will start by defining in Section 1 a generalization of the Jacobian matrix that involves also higher-order derivatives, which is more suitable to this context. Using this matrix, we will give in Section 2 some higher-order criteria of non-singularity. Next, we will prove in Section 3 that the spaces $T_p^n X$ can be identified with the kernel of the higher-order Jacobian, as with the tangent space.

In the last section we will give some applications of the previous results. Firstly, we will generalize a method proposed by D. O'Shea which computes limits of tangent spaces to a singular point of a hypersurface (see [Sh]). This method, along with the theory of Gröbner bases, will allow us to compute examples showing some interesting phenomena of the set of limits of spaces $T_p^n X$. Later, using some results of O. Villamayor appearing in [V], we will explicitly describe the ideal whose blowup gives the higher Nash blowup by means of the higher-order Jacobian matrix.

As a final application, we will study a higher-order version of the following theorem due to A. Nobile: the Nash modification of a variety is an isomorphism if and only if the variety is non-singular (see [No]). We will prove the analogous statement for the higher-order Nash blowup of normal hypersurfaces. To that end, we will show that the singular locus of a hypersurface coincides with the zero set of the ideal whose blowup gives the higher Nash blowup. We will also compute some examples of singular plane curves where the second-order analogue of Nobile's theorem holds as well.

1 A higher-order Jacobian matrix

The first thing we are going to do is to define a higher-order version of the Jacobian matrix of a polynomial. We begin by presenting an example to illustrate the idea of the definition.

Let $F(x, y) = x^3 - y^2 \in \mathbb{C}[x, y]$. Let $p = (a, b) \in X = \mathbf{V}(F)$. The Jacobian matrix of F evaluated at p is defined as:

$$\text{Jac}(F)|_p := (3x^2 \quad -2y)|_p$$

We want to define another matrix involving also higher-order derivatives that generalizes the Jacobian matrix. Let $\mathfrak{a}_p = \langle x - a, y - b \rangle \subset \mathbb{C}[x, y]$. Consider the following linear map:

$$\begin{aligned} \theta : \mathfrak{a}_p &\rightarrow \mathbb{C}^5 \\ f &\mapsto \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{1}{2!} \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{1}{2!} \frac{\partial^2 f}{\partial y^2} \right)|_p. \end{aligned}$$

Let $\mathfrak{b} = \langle F \rangle$. Notice that $\mathfrak{b} \subset \mathfrak{a}_p$. Let $g \cdot F \in \mathfrak{b}$, where $g \in \mathbb{C}[x, y]$. Using repeatedly the Leibniz rule and the fact $F(p) = 0$, we can write $\theta(gF)$ as follows:

$$\begin{aligned} \theta(gF) &= g(p) \cdot (3x^2, -2y, 3x, 0, -1)|_p + \\ &\quad \frac{\partial g}{\partial x}(p) \cdot (F, 0, 3x^2, -2y, 0)|_p + \\ &\quad \frac{\partial g}{\partial y}(p) \cdot (0, F, 0, 3x^2, -2y)|_p. \end{aligned}$$

Let

$$\text{Jac}_2(F) := \begin{pmatrix} 3x^2 & -2y & 3x & 0 & -1 \\ F & 0 & 3x^2 & -2y & 0 \\ 0 & F & 0 & 3x^2 & -2y \end{pmatrix} \quad (1)$$

Thus

$$\theta(gF) = \text{Jac}_2(F)^t|_p \cdot \begin{pmatrix} g(p) \\ \frac{\partial g}{\partial x}(p) \\ \frac{\partial g}{\partial y}(p) \end{pmatrix}$$

We call $\text{Jac}_2(F)$ the *Jacobian matrix of order 2* of F . Now we proceed exactly as in this example to define a higher-order Jacobian of a polynomial. First recall the multi-index notation. Let $\alpha = (\alpha_1, \dots, \alpha_s), \beta = (\beta_1, \dots, \beta_s) \in \mathbb{N}^s$:

- $\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \ \forall i \in \{1, \dots, s\}$.
- $|\alpha| = \alpha_1 + \dots + \alpha_s$.
- $\alpha! = \alpha_1! \cdot \alpha_2! \cdot \dots \cdot \alpha_s!$.
- $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} \dots \binom{\alpha_s}{\beta_s} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.
- $\partial^\alpha = \partial^{\alpha_1} \partial^{\alpha_2} \dots \partial^{\alpha_s}$.

Using this notation, the general Leibniz rule states that

$$\partial^\alpha(g \cdot f) = \sum_{\{\beta|\beta \leq \alpha\}} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \partial^\beta g$$

for any $f, g \in \mathbb{C}[x_1, \dots, x_s]$. If we define $\partial^{\alpha-\beta} f = 0$ when $\alpha_i < \beta_i$ for some $1 \leq i \leq s$, then the general Leibniz rule can also be written as:

$$\partial^\alpha(g \cdot f) = \sum_{\{\beta|0 \leq |\beta| \leq |\alpha|\}} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \partial^\beta g. \quad (2)$$

Let $F \in \mathbb{C}[x_1, \dots, x_s]$ and $p = (a_1, \dots, a_s) \in X = \mathbf{V}(F) \subset \mathbb{C}^s$. Let $\mathfrak{a}_p = \langle x_1 - a_1, \dots, x_s - a_s \rangle \subset \mathbb{C}[x_1, \dots, x_s]$. Fix $n \in \mathbb{N}$. Let $N = \binom{n+s}{s}$ and consider the following linear map:

$$\begin{aligned} \theta : \mathfrak{a}_p &\rightarrow \mathbb{C}^{N-1} \\ f &\mapsto \left(\frac{\partial^\alpha f}{\alpha!} | 1 \leq |\alpha| \leq n \right)_p. \end{aligned} \quad (3)$$

We arrange this vector increasingly using graded lexicographical order, where $\alpha_1 < \alpha_2 < \dots < \alpha_s$.

Let $\mathfrak{b} = \langle F \rangle$. Notice that $\mathfrak{b} \subset \mathfrak{a}_p$. Let $g \cdot F \in \mathfrak{b}$, where $g \in \mathbb{C}[x_1, \dots, x_s]$. Using the general Leibniz rule (2) and the fact $F(p) = 0$, we can write $\theta(gF)$ as follows (recall that we defined $\partial^{\alpha-\beta} F = 0$ if $\alpha_i < \beta_i$ for some i):

$$\theta(gF) = \sum_{\{\beta|0 \leq |\beta| \leq n-1\}} \partial^\beta g(p) \cdot \left(\binom{\alpha}{\beta} \frac{\partial^{\alpha-\beta} F}{\alpha!} | 1 \leq |\alpha| \leq n \right)_p. \quad (4)$$

Let $r_\beta := \beta! \cdot \left(\binom{\alpha}{\beta} \frac{\partial^{\alpha-\beta} F}{\alpha!} | 1 \leq |\alpha| \leq n \right)$, where β is such that $0 \leq |\beta| \leq n-1$. We multiply by $\beta!$ to obtain later some nice properties among these vectors (see lemma 1.1 below). As before, we arrange r_β using graded lexicographical order on α . There are $M = \binom{n+s-1}{s}$ such vectors.

Lemma 1.1. Fix β such that $0 \leq |\beta| \leq n - 1$.

(i) The α -entry of r_β satisfies:

$$r_{\beta,\alpha} = \begin{cases} 0, & \text{if } \alpha_i < \beta_i \text{ for some } 1 \leq i \leq s, \\ \frac{\partial^{\alpha-\beta} F}{(\alpha-\beta)!}, & \text{if } \alpha \geq \beta. \end{cases}$$

In particular, $r_{(0,\dots,0)} = \left(\frac{\partial^\alpha F}{\alpha!} \mid 1 \leq |\alpha| \leq n \right)$.

(ii) Let α be such that $1 \leq |\alpha| \leq n - |\beta|$. Then, $r_{\beta,\beta+\alpha} = r_{(0,\dots,0),\alpha}$.

Now assume $1 \leq |\beta| \leq n - 1$.

(iii) If $\alpha \neq \beta + \alpha'$ for all α' such that $0 \leq |\alpha'| \leq n - |\beta|$ then $r_{\beta,\alpha} = 0$.

(iv) If $\alpha <_{\text{grlex}} \beta + (1, 0, \dots, 0)$ and $\alpha \neq \beta$ then $r_{\beta,\alpha} = 0$.

(v) $r_{\beta,\beta} = F$.

(vi) The only possibly non-zero entries of r_β are those of the form $r_{\beta,\beta+\alpha}$, for some α such that $0 \leq |\alpha| \leq n - |\beta|$. In particular, excepting $r_{\beta,\beta}$, these possibly non-zero entries correspond to shifting by β the α -entries of $r_{(0,\dots,0)}$, where $1 \leq |\alpha| \leq n - |\beta|$, i.e., these entries are (multiples of) the partial derivatives of F of order at most $n - |\beta|$.

Proof. (i) This is just the definition of r_β .

(ii) Indeed, by the hypothesis, $|\beta + \alpha| \leq n$, so it makes sense to consider $r_{\beta,\beta+\alpha}$. Now apply (i).

(iii) Suppose $\alpha \neq \beta + \alpha'$ for all α' such that $0 \leq |\alpha'| \leq n - |\beta|$. We claim that $\alpha_i < \beta_i$ for some $1 \leq i \leq s$. This is clear: if $\alpha_i \geq \beta_i$ for all i , then $\alpha = \beta + (\alpha - \beta)$ and $0 \leq |\alpha - \beta| = |\alpha| - |\beta| \leq n - |\beta|$, which is a contradiction. Therefore, (iii) follows from (i).

(iv) This is a direct consequence of (iii). Indeed, if $\alpha = \beta + \alpha'$ where $1 \leq |\alpha'|$, then $\alpha = \beta + \alpha' \geq_{\text{grlex}} \beta + (1, 0, \dots, 0)$, which contradicts the hypothesis on α .

(v) Indeed, $r_{\beta,\beta} = \frac{\partial^{\beta-\beta} F}{(\beta-\beta)!} = F$.

(vi) This is just a consequence of (ii), (iii), (iv), and (v). □

Definition 1.2. Let $\text{Jac}_n(F)$ be the matrix whose rows are the M vectors r_β . We arrange these rows using graded lexicographical order on β , where $\beta_1 < \beta_2 < \dots < \beta_s$. In particular, $\text{Jac}_n(F)$ is a $(M \times N - 1)$ -matrix. We call $\text{Jac}_n(F)$ the *Jacobian matrix of order n* or the *higher-order Jacobian matrix*.

The higher-order Jacobian matrix satisfies the following properties.

Proposition 1.3. Let $F \in \mathbb{C}[x_1, \dots, x_s]$, $p \in X = \mathbf{V}(F) \subset \mathbb{C}^s$, and $\mathbf{b} = \langle F \rangle$.

- (a) $\text{Jac}_1(F)$ is the usual Jacobian matrix of F .
- (b) $\theta(\mathbf{b}) = \text{Im}(\text{Jac}_n(F)|_p^t)$, where θ was defined in (3) and Im denotes the image of the linear map induced by $\text{Jac}_n(F)|_p^t$.
- (c) Suppose that F is a reduced non-constant polynomial. Suppose $p \in X$ is non-singular and assume $\partial^{(1,0,\dots,0)}F(p) \neq 0$. Under this assumption, $\text{Jac}_n(F)|_p$ is in row echelon form. In addition, every row of $\text{Jac}_n(F)|_p$ has $\partial^{(1,0,\dots,0)}F(p)$ as pivot.

Proof. (a) is immediate by definition of $\text{Jac}_n(F)$. (b) follows from (4) and the fact that for any $(\lambda_1, \dots, \lambda_M) \in \mathbb{C}^M$ there exists $g \in \mathfrak{a}_p$ such that

$$\left(\frac{\partial^\beta g}{\beta!} | 0 \leq |\beta| \leq n-1 \right)_p = (\lambda_1, \dots, \lambda_M).$$

To prove (c), first notice that, for every $0 \leq |\beta| \leq n-1$, $r_{\beta, \beta+(1,0,\dots,0)} = \partial^{(1,0,\dots,0)}F$, according to (ii) of lemma 1.1. Now the fact that $\text{Jac}_n(F)|_p$ is in row echelon form follows from (iv) and (v) of lemma 1.1. \square

2 Higher-order criteria of non-singularity

In this section we will give some criteria of non-singularity using the higher-order Jacobian matrix or some other higher-order data.

2.1 Higher-order version of the Jacobian criterion

Our first goal is to generalize the well-known Jacobian criterion for non-singularity (see [H], Ch. 1, Theorem 5.1). The result is the following:

Theorem 2.1. Let $F \in \mathbb{C}[x_1, \dots, x_s]$ be a reduced non-constant polynomial. Let $p \in X = \mathbf{V}(F) \subset \mathbb{C}^s$. For $n \in \mathbb{N}$, let $M = \binom{n+s-1}{s}$. Then

$$p \text{ is non-singular} \Leftrightarrow \text{rank } \text{Jac}_n(F)|_p = M.$$

Proof. Suppose p is non-singular and assume that $\partial^{(1,0,\dots,0)}F(p) \neq 0$. According to (c) of proposition 1.3, $\text{Jac}_n(F)|_p$ is in row echelon form with $\partial^{(1,0,\dots,0)}F(p)$ as pivots in every row. This implies that the rows of $\text{Jac}_n(F)|_p$ are linearly independent, i.e., $\text{rank } \text{Jac}_n(F)|_p = M$.

Suppose now that $\text{rank } \text{Jac}_n(F)|_p = M$. According to (vi) of lemma 1.1, $r_{(0,\dots,0,n-1)}|_p$ (the last row of $\text{Jac}_n(F)|_p$) contains only first partial derivatives of F as possibly non-zero entries. If all these derivatives evaluated at p were zero then $\text{rank } \text{Jac}_n(F)|_p < M$. Thus, at least one first partial derivative of F evaluated at p is non-zero. We conclude that p is a non-singular point by the usual Jacobian criterion. \square

The previous theorem has the following immediate consequence.

Corollary 2.2. *Let $\mathcal{J}_n \subset \mathbb{C}[x_1, \dots, x_s]/\langle F \rangle$ be the ideal generated by the $(M \times M)$ -minors of $\text{Jac}_n(F)$. Then the singular locus of $X = \mathbf{V}(F)$ corresponds to the zero set of \mathcal{J}_n .*

2.2 Some other higher-order criteria of non-singularity

In this section we prove some other generalizations of well-known results regarding a characterization of non-singularity. We would like to comment that we do not know if the results of this section are particular cases of more general results. On the other hand, the proofs given here are mostly combinatorial.

Lemma 2.3. *Let A be a commutative ring with unity and \mathfrak{m} a maximal ideal of A . Then the natural morphism*

$$\frac{\mathfrak{m}}{\mathfrak{m}^{n+1}} \rightarrow \frac{\mathfrak{m}A_{\mathfrak{m}}}{\mathfrak{m}^{n+1}A_{\mathfrak{m}}}; \quad \bar{f} \mapsto \left[\frac{f}{1} \right],$$

is an isomorphism.

Proof. We proceed by induction. For $n = 1$ it is well known (see, for instance, [L], Chapter 4, Lemma 2.3). Suppose it is true for $n - 1$. Consider the natural homomorphism

$$\begin{aligned} \varphi : \frac{A}{\mathfrak{m}^n} &\rightarrow \frac{A_{\mathfrak{m}}}{\mathfrak{m}^n A_{\mathfrak{m}}}, \\ \bar{a} &\mapsto \left[\frac{a}{1} \right]. \end{aligned}$$

Let $s \in A \setminus \mathfrak{m}$. Then there exist $a \in A$ and $m \in \mathfrak{m}$ such that $as + m = 1$. This implies the following equalities in $A_{\mathfrak{m}}$:

$$\begin{aligned} \frac{a}{1} + \frac{m}{s} &= \frac{1}{s}, \\ \frac{am}{1} + \frac{m^2}{s} &= \frac{m}{s}, \\ &\vdots \\ \frac{am^{n-1}}{1} + \frac{m^n}{s} &= \frac{m^{n-1}}{s}. \end{aligned}$$

But then, modulo $\mathfrak{m}^n A_{\mathfrak{m}}$, we have: $\left[\frac{1}{s}\right] = \left[\frac{a}{1}\right] + \left[\frac{am}{1}\right] + \cdots + \left[\frac{am^{n-1}}{1}\right]$. Thus, $\left[\frac{b}{s}\right] = \left[\frac{ab+abm+\dots+abm^{n-1}}{1}\right]$, which implies that φ is surjective.

Now we show that $\ker \varphi \subset A/\mathfrak{m}^n$ is $\{\bar{0}\}$. $\ker \varphi$ corresponds to some ideal $J \subset A$ satisfying $\mathfrak{m}^n \subset J \subset \mathfrak{m}$. We want to show that $J = \mathfrak{m}^n$. Suppose that there exists $f \in J \setminus \mathfrak{m}^n \subset \mathfrak{m} \setminus \mathfrak{m}^n$. This means that $\bar{0} \neq \bar{f} \in \mathfrak{m}(A/\mathfrak{m}^n)$, but $[f/1] = [0/1] \in \mathfrak{m}(A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}})$. On the other hand, the homomorphism φ restricted to $\mathfrak{m}(A/\mathfrak{m}^n)$ is the natural homomorphism $\mathfrak{m}/\mathfrak{m}^n \rightarrow \mathfrak{m}A_{\mathfrak{m}}/\mathfrak{m}^n A_{\mathfrak{m}}$, which is an isomorphism by the induction hypothesis. This contradicts that $[f/1] = [0/1]$. Therefore φ is an isomorphism. As in [L], Chapter 4, Lemma 2.3, applying the tensor $\otimes_A \mathfrak{m}$ we conclude the proof of the lemma. \square

Lemma 2.4. *Let $\mathfrak{a} \subset \mathbb{C}[x_1, \dots, x_s]$ be a monomial ideal. Assume that there exists $1 \leq i \leq s$ such that all monomials in \mathfrak{a} are multiples of x_i . In other words, assume that $\dim V(\mathfrak{a}) = s - 1$. Let $\mathfrak{m} := \langle x_1, \dots, x_s \rangle \subset \mathbb{C}[x_1, \dots, x_s]/\mathfrak{a}$. Then*

$$\dim_{\mathbb{C}} \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \geq \binom{n+s-2}{s-2}.$$

Proof. Let $l = \min\{\text{total degree of monomials in } \mathfrak{a}\}$. For every $n < l$,

$$\dim_{\mathbb{C}} \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} = \binom{n+s-1}{s-1} > \binom{n+s-2}{s-2},$$

so the lemma is true for these values of n . Now we consider $n = l + j$, $j \geq 0$. Let $L_j := |\{x_1^{\alpha_1} \cdots x_s^{\alpha_s} \in \mathfrak{a} \mid \sum \alpha_i = l + j\}|$, where $j \geq 0$. We claim that

$$\dim_{\mathbb{C}} \frac{\mathfrak{m}^{l+j}}{\mathfrak{m}^{l+j+1}} = \binom{l+j+s-1}{s-1} - L_j. \quad (5)$$

To prove (5) we first observe that the ideal \mathfrak{m}^{l+j} is generated by the (classes of) the elements of the set

$$B := \{x_1^{\alpha_1} \cdots x_s^{\alpha_s} \notin \mathfrak{a} \mid \sum_i \alpha_i = l+j\}.$$

This set has cardinality $\binom{l+j+s-1}{s-1} - L_j$. To show that (the image of) this set is linearly independent in $\mathfrak{m}^{l+j}/\mathfrak{m}^{l+j+1}$, we observe that if there were a non-trivial linear combination of elements of B equal to zero, then we would have $\sum_{x^\alpha \in B} c_\alpha x^\alpha - \sum_{|\beta|=l+j+1} g_\beta x^\beta \in \mathfrak{a}$, for some $c_\alpha \in \mathbb{C}$, $g_\beta \in \mathbb{C}[x_1, \dots, x_s]$, and not all of c_α equal to 0. Thus, for some α , $x^\alpha \in \mathfrak{a}$, since \mathfrak{a} is a monomial ideal. This is a contradiction. Therefore B is linearly independent.

According to the hypothesis on \mathfrak{a} we can assume that the variable x_1 appears in every monomial of \mathfrak{a} . The set of monomials $x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ of total degree $l+j$ such that $\alpha_1 > 0$ has cardinality $\binom{l+j+s-2}{s-1}$. Then (5) concludes the proof of the lemma for these values of n since

$$\binom{l+j+s-1}{s-1} - L_j \geq \binom{l+j+s-1}{s-1} - \binom{l+j+s-2}{s-1} = \binom{l+j+s-2}{s-2}.$$

□

Remark 2.5. The previous lemma is no longer valid if the hypothesis that all monomials in \mathfrak{a} contain one same variable x_i is removed, at least for non-reduced ideals. Let $\mathfrak{a} = \langle x^2, y^2 \rangle$. Then, for $n \geq 3$, $\dim_{\mathbb{C}} \langle x, y \rangle^n / \langle x, y \rangle^{n+1} = 0$.

Corollary 2.6. *Let $X = \mathbf{V}(F) \subset \mathbb{C}^s$, where $F \in \mathbb{C}[x_1, \dots, x_s]$. Assume that $0 \in X$. Let $\mathfrak{m} = \langle x_1, \dots, x_s \rangle \subset \mathbb{C}[x_1, \dots, x_s] / \langle F \rangle$. Then*

$$\dim_{\mathbb{C}} \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \geq \binom{n+s-2}{s-2}.$$

Proof. For this proof, let $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_s]$. Consider any ideal $I \subset \mathbb{C}[x]$ and let $\mathfrak{m} = \langle x_1, \dots, x_s \rangle \subset \mathbb{C}[x]/I$. Let $H_{\mathbb{C}[x]/I}(n) := \dim_{\mathbb{C}} \mathfrak{m}^n / \mathfrak{m}^{n+1}$ be the Hilbert function of $\mathbb{C}[x]/I$. Now denote by F_0 the homogeneous component of F of lowest degree. Let $>$ be any monomial order on $\mathbb{C}[x]$. It is known that the Hilbert functions of $\mathbb{C}[x]/\langle F \rangle$, $\mathbb{C}[x]/\langle F_0 \rangle$, and that of $\mathbb{C}[x]/\langle in_{>}(F_0) \rangle$ coincide (see [E], Theorem 15.26 and Section 15.10.3). Since $\langle in_{>}(F_0) \rangle$ is an ideal generated by a single monomial, we obtain the desired conclusion using lemma 2.4. □

Corollary 2.7. *Let $X = \mathbf{V}(F) \subset \mathbb{C}^s$, where $F \in \mathbb{C}[x_1, \dots, x_s]$. Let $p \in X$ and let \mathfrak{m}_p be its corresponding maximal ideal in $\mathcal{O}_{X,p}$. Then p is non-singular if and only if $\dim_{\mathbb{C}} \mathfrak{m}_p / \mathfrak{m}_p^{n+1} = \binom{n+s-1}{s-1} - 1$.*

Proof. By lemma 2.3, it is enough to prove the statement for $\mathfrak{m} / \mathfrak{m}^{n+1}$, where \mathfrak{m} is the maximal ideal corresponding to p in $\mathbb{C}[x_1, \dots, x_s] / \langle F \rangle$. We proceed by induction on n . Let $n = 1$. If p is non-singular, $\dim_{\mathbb{C}}(\mathfrak{m} / \mathfrak{m}^2) = \dim X = s - 1 = \binom{s}{s-1} - 1$. Now consider the exact sequence of \mathbb{C} -vector spaces:

$$0 \rightarrow \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \rightarrow \frac{\mathfrak{m}}{\mathfrak{m}^{n+1}} \rightarrow \frac{\mathfrak{m}}{\mathfrak{m}^n} \rightarrow 0. \quad (6)$$

Since p is non-singular, $\mathbf{S}^n(\mathfrak{m} / \mathfrak{m}^2) = \mathfrak{m}^n / \mathfrak{m}^{n+1}$, where $\mathbf{S}^n(\cdot)$ denotes the n th-symmetric product. Thus $\dim_{\mathbb{C}}(\mathfrak{m}^n / \mathfrak{m}^{n+1}) = \binom{n+s-2}{s-2}$. On the other hand, by induction, $\dim_{\mathbb{C}}(\mathfrak{m} / \mathfrak{m}^n) = \binom{n+s-2}{s-1} - 1$. By exactness of the sequence, we conclude that

$$\dim_{\mathbb{C}} \left(\frac{\mathfrak{m}}{\mathfrak{m}^{n+1}} \right) = \binom{n+s-2}{s-2} + \left(\binom{n+s-2}{s-1} - 1 \right) = \binom{n+s-1}{s-1} - 1.$$

Now suppose that $p \in X$ is singular. In particular, $\dim_{\mathbb{C}} \mathfrak{m} / \mathfrak{m}^2 > \dim X = s - 1$. Using corollary 2.6 and the exact sequence (6), we conclude by induction that $\dim_{\mathbb{C}} \mathfrak{m} / \mathfrak{m}^{n+1} > \binom{n+s-1}{s-1} - 1$. \square

3 Computing $T_p^n X$

Let $F \in \mathbb{C}[x_1, \dots, x_s]$, $R = \mathbb{C}[x_1, \dots, x_s] / \langle F \rangle$ and $X = \mathbf{V}(F)$. Let $p \in X$ and (R_p, \mathfrak{m}_p) be the localization of R at p . It is well known that the tangent space at p has the following description:

$$T_p X = (\mathfrak{m}_p / \mathfrak{m}_p^2)^\vee = \ker \text{Jac}(F)|_p,$$

where $\text{Jac}(F)$ is the Jacobian matrix of F . The goal of this section is to give an analogous description of the $R_p / \mathfrak{m}_p \cong \mathbb{C}$ -vector space

$$T_p^n X = (\mathfrak{m}_p / \mathfrak{m}_p^{n+1})^\vee,$$

for non-singular points using the higher-order Jacobian. As in previous sections, for $n \in \mathbb{N}$, let $N = \binom{n+s}{s}$ and $M = \binom{n+s-1}{s}$.

Remark 3.1. Notice that for non-singular points of a hypersurface $p \in X \subset \mathbb{C}^s$, $T_p X$ is a hyperplane in \mathbb{C}^s . However, for $n > 1$ the space $T_p^n X \hookrightarrow T_p^n \mathbb{C}^s \cong \mathbb{C}^{N-1}$ is not a hyperplane of \mathbb{C}^{N-1} (see corollary 2.7).

Lemma 3.2. *Let $p \in X \subset \mathbb{C}^s$. Fix $n \in \mathbb{N}$ and let $\text{Jac}_n(F)$ be the higher-order Jacobian matrix of F . Then we have the following identification:*

$$T_p^n X \cong \left(\frac{\mathbb{C}^{N-1}}{\text{Im}(\text{Jac}_n(F)|_p^t)} \right)^\vee.$$

Proof. The proof consists in adapting the usual proof for the case $n = 1$ to the case $n > 1$ (see for instance [H], Chapter I, Theorem 5.1). Let $p = (a_1, \dots, a_s)$ and $\mathfrak{a}_p = \langle x_1 - a_1, \dots, x_s - a_s \rangle \subset \mathbb{C}[x_1, \dots, x_s]$. As in section 1, consider the following linear map:

$$\begin{aligned} \theta : \mathfrak{a}_p &\rightarrow \mathbb{C}^{N-1} \\ f &\mapsto \left(\frac{\partial^\alpha f}{\alpha!} \mid 1 \leq |\alpha| \leq n \right)_p. \end{aligned}$$

This map is surjective since $(x - a)^\alpha$, for $1 \leq |\alpha| \leq n$, are mapped to the canonical basis of \mathbb{C}^{N-1} . On the other hand, by observing the Taylor expansion of an element of \mathfrak{a}_p around p we see that $\ker \theta = \mathfrak{a}_p^{n+1}$. Thus,

$$\frac{\mathfrak{a}_p}{\mathfrak{a}_p^{n+1}} \cong \mathbb{C}^{N-1}. \quad (7)$$

Let $\mathfrak{b} = \langle F \rangle$. According to (b) of proposition 1.3, $\theta(\mathfrak{b}) = \text{Im}(\text{Jac}_n(F)|_p^t)$. Using the isomorphism (7) we also have $\theta(\mathfrak{b}) \cong \mathfrak{b}(\mathfrak{a}_p/\mathfrak{a}_p^{n+1}) = (\mathfrak{b} + \mathfrak{a}_p^{n+1})/\mathfrak{a}_p^{n+1}$. Then

$$\frac{\frac{\mathfrak{a}_p}{\mathfrak{a}_p^{n+1}}}{\frac{\mathfrak{b} + \mathfrak{a}_p^{n+1}}{\mathfrak{a}_p^{n+1}}} \cong \frac{\mathfrak{a}_p}{\mathfrak{b} + \mathfrak{a}_p^{n+1}}.$$

Since $\mathfrak{b} \subset \mathfrak{a}_p$, it follows that

$$\frac{\mathfrak{a}_p \left(\frac{\mathbb{C}[x_1, \dots, x_s]}{\mathfrak{b}} \right)}{\mathfrak{a}_p^{n+1} \left(\frac{\mathbb{C}[x_1, \dots, x_s]}{\mathfrak{b}} \right)} \cong \frac{\frac{\mathfrak{a}_p + \mathfrak{b}}{\mathfrak{b}}}{\frac{\mathfrak{a}_p^{n+1} + \mathfrak{b}}{\mathfrak{b}}} = \frac{\frac{\mathfrak{a}_p}{\mathfrak{b}}}{\frac{\mathfrak{a}_p^{n+1} + \mathfrak{b}}{\mathfrak{b}}} \cong \frac{\mathfrak{a}_p}{\mathfrak{a}_p^{n+1} + \mathfrak{b}}.$$

By identifying (see lemma 2.3) $\frac{\mathfrak{m}_p}{\mathfrak{m}_p^{n+1}} \cong \frac{\mathfrak{a}_p \left(\frac{\mathbb{C}[x_1, \dots, x_s]}{\mathfrak{b}} \right)}{\mathfrak{a}_p^{n+1} \left(\frac{\mathbb{C}[x_1, \dots, x_s]}{\mathfrak{b}} \right)}$, we conclude

$$T_p^n X = \left(\frac{\mathfrak{m}_p}{\mathfrak{m}_p^{n+1}} \right)^\vee \cong \left(\frac{\frac{\mathfrak{a}_p}{\mathfrak{a}_p^{n+1}}}{\frac{\mathfrak{b} + \mathfrak{a}_p^{n+1}}{\mathfrak{a}_p^{n+1}}} \right)^\vee \cong \left(\frac{\mathbb{C}^{N-1}}{\text{Im}(\text{Jac}_n(F)|_p^t)} \right)^\vee.$$

□

Now assume that $p \in X$ is a non-singular point. We claim that

$$\left(\frac{\mathbb{C}^{N-1}}{\text{Im}(\text{Jac}_n(F)|_p^t)} \right)^\vee \cong \ker \text{Jac}_n(F)|_p. \quad (8)$$

To prove this, let $e_i^\vee : \mathbb{C}^{N-1} \rightarrow \mathbb{C}$, $t = (t_1, \dots, t_{N-1}) \mapsto t_i$. Then

$$\ker \text{Jac}_n(F)|_p \hookrightarrow \{ \phi : \mathbb{C}^{N-1} \rightarrow \mathbb{C} \mid \text{Im}(\text{Jac}_n(F)|_p^t) \subset \ker \phi \} \quad (9)$$

via the map $t \mapsto \sum_i t_i e_i^\vee$. Since p is a non-singular point, theorem 2.1 implies $\dim \ker \text{Jac}_n(F)|_p = N - M - 1$. Now, since $(\mathbb{C}^{N-1}/\text{Im}(\text{Jac}_n(F)|_p^t))^\vee \cong \{ \phi : \mathbb{C}^{N-1} \rightarrow \mathbb{C} \mid \text{Im}(\text{Jac}_n(F)|_p^t) \subset \ker \phi \}$, corollary 2.7 and lemma 3.2 imply: $\dim \{ \phi : \mathbb{C}^{N-1} \rightarrow \mathbb{C} \mid \text{Im}(\text{Jac}_n(F)|_p^t) \subset \ker \phi \} = \dim_{\mathbb{C}}(\mathfrak{m}_p/\mathfrak{m}_p^{n+1})^\vee = N - M - 1$. Therefore the inclusion (9) is actually an equality. This proves claim (8). Using lemma 3.2 we conclude:

Proposition 3.3. *Let $F \in \mathbb{C}[x_1, \dots, x_s]$ and $p \in X = \mathbf{V}(F) \subset \mathbb{C}^s$ be a non-singular point. Then $T_p^n X = \ker \text{Jac}_n(F)|_p$.*

4 Some applications

In this final section we will give some applications of the constructions and results of previous sections. Firstly, we will generalize a result of O'Shea appearing in [Sh] that computes limits of tangent spaces to singular points of a hypersurface. Secondly, applying some results of Villamayor appearing in [V], we will describe an ideal whose blow up is the higher Nash blowup of a hypersurface. Using this ideal, we will prove a higher-order analogue of Nobile's theorem for normal hypersurfaces.

4.1 Limits of $T_p^n X$, where X is a hypersurface

We start by revisiting a theorem due to D. O'Shea appearing in [Sh] which gives a method to compute limits of tangent spaces to a singular point of a hypersurface. We will see that this result is still valid if we replace tangent space by $T_p^n X$, for any $n \in \mathbb{N}$, essentially with the same proof. This theorem will allow us to compute the space of limits of $T_p^n X$ using the theory of Gröbner bases. In particular, this method provides a way to compute the fibers of the higher Nash blowup of a hypersurface.

Definition 4.1. Let $X = \mathbf{V}(F) \subset \mathbb{C}^s$ where $F \in \mathbb{C}[x_1, \dots, x_s]$, and let $S(X)$ denotes the singular locus of X . Assume that $0 \in X$. The space of limits of $T_p^n X$ at 0 is the set

$$\{T \in Gr(N - M - 1, \mathbb{C}^{N-1}) | \exists \{p_k\} \subset X \setminus S(X) \text{ s.t. } p_k \rightarrow 0 \text{ and } T_{p_k}^n X \rightarrow T\},$$

where $Gr(N - M - 1, \mathbb{C}^{N-1})$ denotes the Grassmanian of vector spaces of dimension $N - M - 1$ in \mathbb{C}^{N-1} . We denote the space of limits of $T_p^n X$ as $\mathcal{L}_n(X, 0)$. By using Plücker coordinates, we embed $Gr(N - M - 1, \mathbb{C}^{N-1})$ in a projective space so, when we mention the space $T_p^n X$ or a limit of such, we consider them as points in such a projective space.

Remark 4.2. We will use the duality between $Gr(N - M - 1, \mathbb{C}^{N-1})$ and $Gr(M, \mathbb{C}^{N-1})$ to compute $\mathcal{L}_n(X, 0)$. More precisely, in the next theorem we will compute limits of vector spaces of dimension M defined as the span of the rows of $\text{Jac}_n(F)|_{p_k}$, where $\{p_k\} \subset X$ is a sequence of non-singular points converging to 0 (recall that $\text{rank}(\text{Jac}_n(F)|_{p_k}) = M$ in this case). By duality, we will obtain the set $\mathcal{L}_n(X, 0)$.

Let $\Lambda = \{(\alpha_1, \dots, \alpha_M) | \alpha_i \in \mathbb{N}^s, 1 \leq |\alpha| \leq n, \alpha_1 <_{grlex} \dots <_{grlex} \alpha_M\}$. For $J \in \Lambda$, denote by Δ_J the determinant of the matrix formed by the M columns of $\text{Jac}_n(F)$ corresponding to J .

Theorem 4.3. (cf. [Sh], Proposition 1) Let $X = \mathbf{V}(F) \subset \mathbb{C}^s$ be a hypersurface, where $F \in \mathbb{C}[x_1, \dots, x_s]$ and assume that $0 \in X$ is a singular point. Consider the following ideal in $\mathbb{C}[x_1, \dots, x_s, t, u_J | J \in \Lambda]$:

$$A = \langle F, u_J - t\Delta_J | J \in \Lambda \rangle.$$

Then $\mathcal{L}_n(X, 0)$ can be identified with the variety

$$\mathbf{V}\left(\frac{A \cap \mathbb{C}[x_1, \dots, x_s, u_J | J \in \Lambda]}{\langle x_1, \dots, x_s \rangle}\right).$$

Proof. Make $(x, u) = ((x_1, \dots, x_s), (u_J | J \in \Lambda))$. The idea of the proof consists in showing that points in $\mathbf{V}(A \cap \mathbb{C}[x, u])$ represent points $x \in X$ along with complex multiples of $(\Delta_J(x) | J \in \Lambda)$ or limits of such.

Suppose first that $x_0 \in X$ is non-singular. We claim that $(x_0, u_0) \in \mathbf{V}(A \cap \mathbb{C}[x, u])$ if and only if u_0 is a complex multiple of $(\Delta_J(x_0) | J \in \Lambda)$. Let $(x_0, u_0) \in \mathbf{V}(A \cap \mathbb{C}[x, u])$. Since x_0 is non-singular, by theorem 2.1, $\Delta_J(x_0) \neq 0$ for some $J \in \Lambda$. In particular, $(x_0, u_0) \notin \mathbf{V}(F, \Delta_J | J \in \Lambda)$. According to [CLO],

Ch. 3, Section 1, Theorem 3, the partial solution $(x_0, u_0) \in \mathbf{V}(A \cap \mathbb{C}[x, u])$ extends to a solution $(x_0, t_0, u_0) \in \mathbf{V}(A)$, for some $t_0 \in \mathbb{C}$. This implies that $u_0 - t_0 \Delta_J(x_0) = 0$ for all $J \in \Lambda$, i.e., $u_0 = t_0(\Delta_J(x_0) | J \in \Lambda)$. Suppose now that u_0 is a complex multiple of $(\Delta_J(x_0) | J \in \Lambda)$. In particular, $(x_0, t, u_0) \in \mathbf{V}(A)$, for some $t \in \mathbb{C}$. This immediately implies that $(x_0, u_0) \in \mathbf{V}(A \cap \mathbb{C}[x, u])$ (see [CLO], Ch. 3, Section 2, Lemma 1).

Now we suppose that $x_0 \in X$ is singular. We claim that $(x_0, u_0) \in \mathbf{V}(A \cap \mathbb{C}[x, u])$ if and only if u_0 is limit of multiples of $(\Delta_J(x) | J \in \Lambda)$ for a sequence of non-singular points converging to x_0 .

We start with the second implication. Let $\{x_k\} \subset X \setminus S(X)$ be a sequence such that $x_k \rightarrow x_0$ and let $\{u_k\}$ be the sequence of multiples of $(\Delta_J(x_k) | J \in \Lambda)$ converging to u_0 . By the non-singular case we have that $(x_k, u_k) \in \mathbf{V}(A \cap \mathbb{C}[x, u])$ for all k . Then $(x_k, u_k) \rightarrow (x_0, u_0)$ and since $\mathbf{V}(A \cap \mathbb{C}[x, u])$ is a closed set then we must have $(x_0, u_0) \in \mathbf{V}(A \cap \mathbb{C}[x, u])$.

Let us suppose now that $(x_0, u_0) \in \mathbf{V}(A \cap \mathbb{C}[x, u])$. Since x_0 is singular, $\Delta_J(x_0) = 0$ for all $J \in \Lambda$, according to theorem 2.1. On the other hand, we can assume $u_0 \neq 0$ (if $u_0 = 0$ the claim is trivially true). These facts imply $(x_0, t, u_0) \notin \mathbf{V}(A)$ for all $t \in \mathbb{C}$. Therefore, $(x_0, u_0) \notin \pi_t(\mathbf{V}(A))$, where π_t is the projection to the x and u coordinates. According to [CLO], Ch. 3, Section 2, Theorem 3, we know that $\overline{\pi_t(\mathbf{V}(A))} = \mathbf{V}(A \cap \mathbb{C}[x, u])$. Since $(x_0, u_0) \in \mathbf{V}(A \cap \mathbb{C}[x, u])$ it follows that (x_0, u_0) is limit of points in $\pi_t(\mathbf{V}(A))$ (notice that we are using the fact that topological and algebraic closure coincides), i.e., $(x_k, u_k) \rightarrow (x_0, u_0)$ for some sequence $\{(x_k, u_k)\} \subset \pi_t(\mathbf{V}(A))$. Thus, there exists $\{(x_k, t_k, u_k)\} \subset \mathbf{V}(A)$ such that $\pi_t(x_k, t_k, u_k) = (x_k, u_k)$. In particular, u_k is a complex multiple of $(\Delta_J(x_k) | J \in \Lambda)$. If $x_k \in X$ is singular for all k then $u_k = 0$, so that $(x_k, 0) = \pi_t(x_k, t_k, 0)$ is such that $(x_k, 0) \rightarrow (x_0, u_0) \neq (x_0, 0)$, which is a contradiction. We conclude that there are at most a finite number of singular points in $\{x_k\}$. Taking k sufficiently large, we have a non-singular sequence. This finishes the proof of the claim.

To conclude the proof of the theorem we notice the following natural bijective correspondence: $\mathbf{V}(A \cap \mathbb{C}[x, u]) \cap \{x = 0\} \leftrightarrow \mathbf{V}((A \cap \mathbb{C}[x, u])/\langle x \rangle)$. Thus, points in $\mathbf{V}((A \cap \mathbb{C}[x, u])/\langle x \rangle)$ correspond to limits of complex multiples of vectors $(\Delta_J(x_k) | J \in \Lambda)$, where $\{x_k\} \subset X \setminus S(X)$ and $x_k \rightarrow 0$. These vectors determine the spaces $T_{x_k}^n X$. We have obtained the desired identification. \square

Next we present a simple example to illustrate the method of the previous theorem.

Example 4.4. Let $F = x^3 - y^2$ and $X = \mathbf{V}(F) \subset \mathbb{C}^2$. After computing the corresponding minors of $\text{Jac}_2(F)$ (see (1)) we define:

$$\begin{aligned} A = \langle & F, u_1 - (3xF - 9x^4)tF, u_2 - (12x^2y)tF, u_3 + (4y^2 + F)tF, \\ & u_4 - 27x^6t + (9x^3)tF, u_5 + 18x^4yt + (6xy)tF, u_6 - 12x^2y^2t - (3x^2)tF, \\ & u_7 + 18x^4yt - (6xy)tF, u_8 - 12x^2y^2t + (3x^2)tF, u_9 + 8y^3t + (2y)tF, \\ & u_{10} - (12xy^2 - 9x^4)t \rangle. \end{aligned}$$

A is an ideal in $\mathbb{C}[x, y, t, u_1, \dots, u_{10}]$. Now we use the theory of Gröbner bases to compute a basis of $A \cap \mathbb{C}[x, y, u_1, \dots, u_{10}] / \langle x, y \rangle$. First, using SINGULAR 3-1-6 ([DGPS]), we compute a Gröbner basis of A with respect to lexicographical order assuming $t > x > y > u_i$, call it G . Then $G \cap \mathbb{C}[x, y, u_1, \dots, u_{10}]$ is a basis of $A \cap \mathbb{C}[x, y, u_1, \dots, u_{10}]$ (see [CLO], Ch. 3, Section 1, Theorem 2). By making $x = 0, y = 0$ in the resulting set we obtain $A \cap \mathbb{C}[x, y, u_1, \dots, u_{10}] / \langle x, y \rangle = \langle u_1, u_2, u_3, u_4, u_5, u_6^2, u_8, u_9, u_{10} \rangle$. It follows that the zero set of this ideal is $L = \{(0, \dots, 0, a_7, 0, 0, 0) \in \mathbb{C}^{10}\}$. This means that $\mathcal{L}_n(X, 0)$ consists of only one limit of spaces $T_p^2 X$, for any sequence of non-singular points converging to the origin, which corresponds to the projectivization of L .

Example 4.5. Let $F = x^3 + x^2 - y^2$ and $X = \mathbf{V}(F) \subset \mathbb{C}^2$. Consider the Jacobian matrix of order 2 of F :

$$\text{Jac}_2(F) := \begin{pmatrix} 3x^2 + 2x & -2y & 3x + 1 & 0 & -1 \\ F & 0 & 3x^2 + 2x & -2y & 0 \\ 0 & F & 0 & 3x^2 + 2x & -2y \end{pmatrix}$$

As in the previous example we find that a basis of $A \cap \mathbb{C}[x, y, u_1, \dots, u_{10}] / \langle x, y \rangle$ is given by the set $\{u_1 - 2u_7, u_2 - u_3, u_3 - u_6, u_4 - u_5, u_5 - 2u_7, u_6^2 - 4u_7^2, u_8, u_9, u_{10}\}$. The zero set of this ideal is the following set:

$$\begin{aligned} L = \{ & (a_1, a_2, \dots, a_{10}) \in \mathbb{C}^{10} \mid a_1 = a_4 = a_5 = 2a_7, a_2 = a_3 = a_6, \\ & a_2^2 - 4a_7^2 = 0, a_8 = a_9 = a_{10} = 0 \}. \end{aligned}$$

In particular, there are only two different limits of spaces $T_p^2 X$ corresponding to the projectivization of L .

The higher Nash blowup is a modification of a variety. In particular, for curves, its fibers are finite sets. This is not necessarily true for varieties of higher dimension as the following example shows.

Example 4.6. Let $F = xy - z^4$ and $X = \mathbf{V}(F) \subset \mathbb{C}^3$. It is well known that $\mathcal{L}_1(X, 0)$ is an infinite set: any plane in \mathbb{C}^3 containing the z -axis is a limit of tangent spaces (see the example following Proposition 1 in [Sh]). Now we show that $\mathcal{L}_2(X, 0)$ is also infinite. Consider the Jacobian matrix of order 2 of F :

$$\text{Jac}_2(F) := \begin{pmatrix} y & x & -4z^3 & 0 & 1 & 0 & 0 & 0 & -6z^2 \\ F & 0 & 0 & y & x & 0 & -4z^3 & 0 & 0 \\ 0 & F & 0 & 0 & y & x & 0 & -4z^3 & 0 \\ 0 & 0 & F & 0 & 0 & 0 & y & x & -4z^3 \end{pmatrix}$$

After carefully computing the minors of $\text{Jac}_2(F)$, a basis for the ideal $A \cap \mathbb{C}[x, y, z, u_1, \dots, u_{126}]/\langle x, y, z \rangle$ is given by the following set:

$$\begin{aligned} &\{u_1, \dots, u_{37}, u_{39}, \dots, u_{82}, u_{84}, \dots, u_{111}, u_{114}^2, u_{115}^3, u_{116}^2, u_{117}, \dots, u_{121}, \\ &u_{122}^2, u_{123}, u_{124}^2, u_{125}, u_{126}, u_{38}u_{83}, u_{38}u_{113}, u_{38}u_{114}, u_{38}u_{122}, u_{38}u_{124}, \\ &u_{83}u_{112}, u_{83}u_{114}, u_{83}u_{115}, u_{83}u_{116}, u_{112}u_{124}, u_{113}u_{115}^2, u_{113}u_{116}, \\ &u_{114}u_{115}, u_{114}u_{116}, u_{114}u_{122}, u_{114}u_{124}, u_{115}u_{116}, u_{115}u_{122}, u_{115}u_{124}, \\ &u_{116}u_{122}, u_{116}u_{124}, u_{122}u_{124}, u_{112}u_{114} + u_{113}u_{115}, u_{112}u_{122} - u_{113}u_{114}, \\ &8u_{112}u_{116} + 3u_{115}^2, 8u_{113}u_{124} + 3u_{122}^2\}. \end{aligned}$$

The zero set of this ideal in \mathbb{C}^{126} is:

$$\begin{aligned} L = \{(a_1, a_2, \dots, a_{126}) \in \mathbb{C}^{126} \mid &a_1 = \dots = a_{37} = a_{39} = \dots = a_{82} = a_{84} = 0, \\ &a_{85} = \dots = a_{111} = a_{114} = \dots = a_{126} = 0, \\ &a_{38}a_{83} = a_{38}a_{113} = a_{83}a_{112} = 0\}. \end{aligned}$$

L corresponds to three 2-dimensional planes in \mathbb{C}^{126} : $P_1 = \text{span}\{e_{112}, e_{113}\}$, $P_2 = \text{span}\{e_{38}, e_{112}\}$, $P_3 = \text{span}\{e_{83}, e_{113}\}$. After projectivization we obtain three lines in \mathbb{P}^{125} , call them l_1, l_2, l_3 . These l_i give place to the following families of 4-dimensional vector spaces of \mathbb{C}^9 :

$$\begin{aligned} &\{(0, 0, 0, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, 0) \in \mathbb{C}^9 \mid a\lambda_8 - b\lambda_7 = 0, (a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}\}, \\ &\{(\lambda_1, 0, 0, \lambda_4, \lambda_5, \lambda_6, \lambda_7, 0, 0) \in \mathbb{C}^9 \mid a\lambda_1 - d\lambda_6 = 0, (a, d) \in \mathbb{C}^2 \setminus \{(0, 0)\}\}, \\ &\{(0, \lambda_2, 0, \lambda_4, \lambda_5, \lambda_6, 0, \lambda_8, 0) \in \mathbb{C}^9 \mid b\lambda_2 - c\lambda_4 = 0, (b, c) \in \mathbb{C}^2 \setminus \{(0, 0)\}\}, \end{aligned}$$

respectively. Taking orthogonal complements, we have that any 5-dimensional vector space $W \subset \mathbb{C}^9$ such that $W \subset \text{span}(e_1, e_2, e_3, e_7, e_8, e_9)$ and contains $\text{span}(e_1, e_2, e_3, e_9)$, or $W \subset \text{span}(e_1, e_2, e_3, e_6, e_8, e_9)$ and contains $\text{span}(e_2, e_3, e_8, e_9)$, or $W \subset \text{span}(e_1, e_2, e_3, e_4, e_7, e_9)$ and contains $\text{span}(e_1, e_3, e_7, e_9)$, is a limit of spaces $T_p^2 X$, where e_i denotes the canonical basis of \mathbb{C}^9 .

4.2 An ideal defining the higher Nash blowup of a hypersurface

The goal of this section is to prove that the ideal whose blowup is the higher Nash blowup of order n of a hypersurface, correspond to the ideal generated by the maximal minors of the Jacobian matrix of order n of the polynomial defining the hypersurface. This fact will be a direct consequence of a more general result of O. Villamayor appearing in [V]. With this ideal at hand, we prove that the higher-order version of Nobile's theorem is true for normal hypersurfaces. We also exhibit some examples of singular plane curves where this result holds for the higher Nash blowup of order 2.

Let us first expose the results we are going to need. Let $F \in \mathbb{C}[x_1, \dots, x_s]$ be an irreducible polynomial, $R = \mathbb{C}[x_1, \dots, x_s]/\langle F \rangle$ and $X = \text{Spec } R$. Let $I := \ker(R \otimes R \rightarrow R)$. We give structure of R -module to I via the map $R \rightarrow R \otimes R$, $r \mapsto r \otimes 1$. Let K be the field of fractions of R and let $r = \dim_K I/I^{n+1} \otimes_R K$ be the generic rank of I/I^{n+1} . Consider the following fractionary ideal of K :

$$\mathfrak{b} := \text{Im}\left(\bigwedge^r \frac{I}{I^{n+1}} \rightarrow \bigwedge^r \frac{I}{I^{n+1}} \otimes_R K \cong K\right).$$

Theorem 4.7. *The higher Nash blowup of X is isomorphic to the blowup of the fractionary ideal \mathfrak{b} .*

Proof. See [Y], Proposition 1.8, and [OZ], Theorem 3.1. \square

The ideal \mathfrak{b} can be explicitly described as follows. Consider a presentation of the module I/I^{n+1} by a $(\Lambda \times \Lambda')$ -matrix A :

$$R^{\Lambda'} \xrightarrow{A} R^{\Lambda} \longrightarrow \frac{I}{I^{n+1}} \longrightarrow 0. \quad (10)$$

Then there exist $\Lambda - r$ columns of A such that the $\Lambda \times (\Lambda - r)$ -matrix A' formed by these columns has rank $\Lambda - r$.

Proposition 4.8. *The ideal $\mathcal{J}_n \subset R$ generated by the $(\Lambda - r)$ -minors of A' is equal to \mathfrak{b} for a suitable choice of isomorphism $\bigwedge^r \frac{I}{I^{n+1}} \otimes_R K \cong K$. In addition, the ideal \mathcal{J}_n is independent of the choice of the $\Lambda - r$ columns of A as long as the rank of the matrix formed by these columns is $\Lambda - r$.*

Proof. This is a particular case of [V], Proposition 2.5 and Corollary 2.6. For $n = 1$, this is Theorem 1 of [No] or Theorem 1 of [GS-1] (Section 2). \square

With these results at hand, now we can look for the ideal defining the higher Nash blowup. It is well known that $I = \langle x_i \otimes 1 - 1 \otimes x_i \mid i = 1, \dots, s \rangle$. Now consider the following isomorphisms of rings (let $x = (x_1, \dots, x_s)$, $x' = (x'_1, \dots, x'_s)$):

$$\begin{aligned} R \otimes_{\mathbb{C}} R &\cong \mathbb{C}[x, x'] / \langle F(x), F(x') \rangle \cong \mathbb{C}[x, x' - x] / \langle F(x), F(x') \rangle \\ &\quad (\text{let } \Delta x := x' - x) \\ &\cong \mathbb{C}[x, \Delta x] / \langle F(x), F(x + \Delta x) \rangle \\ &= \mathbb{C}[x, \Delta x] / \langle F(x), \sum_{|\alpha| \geq 1} \frac{\partial^\alpha F}{\alpha!} (\Delta x)^\alpha \rangle. \end{aligned}$$

In this isomorphic ring, $I = \langle \Delta x_1, \dots, \Delta x_s \rangle$. Thus, the quotient of R -modules I/I^{n+1} is generated by $\{[(\Delta x)^\alpha] \mid 1 \leq |\alpha| \leq n\}$. This set has cardinality $N - 1$ (recall that $N = \binom{n+s}{s}$).

Let $\{e_\alpha \mid 1 \leq |\alpha| \leq n\}$ denotes the canonical basis of R^{N-1} (we arrange the set of such α increasingly by graded lexicographical order assuming $\alpha_1 < \dots < \alpha_s$). Consider the following surjective map, $\theta : R^{N-1} \rightarrow I/I^{n+1}$, $e_\alpha \mapsto [(\Delta x)^\alpha]$. Viewing the rows r_β of $\text{Jac}_n(F)$ as elements of R^{N-1} (so the entries of $\text{Jac}_n(F)$ are taken modulo F), we notice that (see (i) of lemma 1.1):

$$\begin{aligned} \theta(r_\beta) &= \left[\sum_{1 \leq |\alpha| \leq n} \left(\beta! \binom{\alpha}{\beta} \frac{\partial^{\alpha-\beta} F}{\alpha!} \right) (\Delta x)^\alpha \right] \\ &= \left[\sum_{1 \leq |\alpha| \leq n, \alpha > \beta} \frac{\partial^{\alpha-\beta} F}{(\alpha - \beta)!} (\Delta x)^\alpha \right] \\ &= \left[(\Delta x)^\beta \left(\sum_{1 \leq |\alpha| \leq n, \alpha > \beta} \frac{\partial^{\alpha-\beta} F}{(\alpha - \beta)!} (\Delta x)^{\alpha-\beta} \right) \right] \\ &= \left[(\Delta x)^\beta \left(- \sum_{|\alpha| > n-|\beta|} \frac{\partial^\alpha F}{\alpha!} (\Delta x)^\alpha \right) \right] = [0], \end{aligned} \tag{11}$$

where the last equality follows from the fact that every element $(\Delta x)^{\alpha+\beta}$ appearing on the sum satisfies $|\alpha| + |\beta| > n - |\beta| + |\beta| = n$. In particular, every row of $\text{Jac}_n(F)$ represents a relation of the generators of I/I^{n+1} .

Lemma 4.9. *The generic rank of I/I^{n+1} is $\binom{n+s-1}{s-1} - 1$.*

Proof. This is just the local version of known results on the sheaf of principal parts (see [G], Paragraph 16.3.7 and [LT], Section 4). \square

Proposition 4.10. *The ideal \mathcal{J}_n defining the higher Nash blowup of X coincides with the ideal generated by the maximal minors of $\text{Jac}_n(F)$.*

Proof. Consider the following presentation of I/I^{n+1} :

$$R^{\Lambda'} \xrightarrow{A} R^{N-1} \xrightarrow{\theta} \frac{I}{I^{n+1}} \longrightarrow 0.$$

According to (11), we have that $\text{Jac}_n(F)^t$ is a submatrix of A . After change of coordinates if necessary, we can assume that $\partial^{(1,0,\dots,0)} F \neq 0$. As in proposition 1.3, it follows that $\text{Jac}_n(F)$ is in row echelon form. Thus $\text{rank } \text{Jac}_n(F) = M$ (recall that $M = \binom{n+s-1}{s}$). On the other hand, since $M = N-1 - (\binom{n+s-1}{s-1} - 1)$, the previous lemma implies that $\text{Jac}_n(F)^t$ satisfies the requirements on the submatrix A' of A in (10). By proposition 4.8, we conclude that the ideal whose blowup gives the higher Nash blowup of X , coincides with the ideal generated by the maximal minors of $\text{Jac}_n(F)$. \square

To give an example of how can we use the explicit description of the ideal defining the higher-order Nash blowup of a hypersurface, we are going to study a higher-order version of the following theorem due to A. Nobile.

Theorem 4.11. *Let X be an equidimensional algebraic variety over \mathbb{C} . Let (X^*, ν) be the Nash modification of X . Then ν is an isomorphism if and only if X is non-singular.*

Proof. See [No], Theorem 2. \square

We can naturally ask if this theorem holds when we replace Nash modification by the higher Nash blowup. The next example considers the case of the second Nash blowup of some singular plane curves.

Example 4.12. Let $F = y^p - x^q \in \mathbb{C}[x, y]$, where $2 \leq p < q$ and $(p, q) = 1$. Let $X = \mathbf{V}(F) \subset \mathbb{C}^2$. After computing the maximal minors of $\text{Jac}_2(F)$, we obtain that the ideal \mathcal{J}_2 of proposition 4.10 is generated by (recall that we take the minors modulo F):

$$\mathcal{J}_2 = \begin{cases} \langle x^{q-2}y^{2p-2}, y^{3p-3} \rangle, & \text{if } p = 2, 3, \\ \langle x^{q-3}y^{2p}, x^{q-2}y^{2p-2}, y^{3p-3} \rangle, & \text{if } p > 3. \end{cases}$$

Notice that \mathcal{J}_2 is a non-principal ideal in every case (this can be seen by using the isomorphism $\mathbb{C}[x, y]/\langle F \rangle \cong \mathbb{C}[u^p, u^q]$ and the hypothesis on p, q). In particular, the higher Nash blowup of order 2 of X is not an isomorphism (see [L], Chapter 8, Proposition 1.12). Since X is a singular curve, this shows that the analogue of Nobile's theorem on the usual Nash blowup is also true for the higher Nash blowup of order 2 for this family of curves.

Even though the strategy in the previous example is unlikely to work for the general case of a hypersurface, we can still use proposition 4.10 to show that the higher-order analogue of Nobile’s theorem holds for normal hypersurfaces.

Theorem 4.13. *Let $F \in \mathbb{C}[x_1, \dots, x_s]$ be an irreducible polynomial and $X = V(F) \subset \mathbb{C}^s$. Suppose X is normal. Let $(\text{Nash}_n(X), \pi_n)$ be the higher Nash blowup of order n of X . Then π_n is an isomorphism if and only if X is non-singular.*

Proof. π_n only modifies singular points so if X is non-singular then π_n is an isomorphism. Suppose now that X is singular. Let \mathcal{J}_n be the ideal defining the higher Nash blowup of order n of X . According to proposition 4.10 and corollary 2.2, the zero set of \mathcal{J}_n coincides with the singular locus $S(X)$ of X . Since X is normal, $\dim S(X) \leq d - 2$, where $d = \dim X$. It follows that \mathcal{J}_n must be generated by at least two elements. Now we use the fact that the blowup of a non-principal ideal is not an isomorphism. \square

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